Ordered Cubes

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Motivation

Combinatorial presentations of the simplex category typically characterize its objects in terms of totally ordered inhabited finite sets. From this perspective, although an n-simplex is regarded as having structure in all dimensions up to n, this structure is somehow generated by the lowest two dimensions: the 0-dimensional vertices being ordered, and the 1-dimensional edges doing the ordering. Maps between simplices are described in terms of maps of the 0-dimensional structure that preserve the 1-dimensional structure. From this perspective, the simplex category can be thought of as being specified "bottom-up". In contrast, combinatorial presentations of cube categories tend to be specified in a "top-down" manner, with maps out of an n-cube determining certain cubes of adjacent dimensions, and only transitively determining vertices.

Inspired by the "bottom-up" interpretation of the simplex category, we present a vertex-based cube category, which includes the maps familiar from other cube categories (faces, degeneracies, diagonals, and connections), but which contains other potentially interesting maps as well. This *ordered cube category* is described in terms of monotone maps of preordered finite sets, so enjoys computational properties such as decidability of equality. It is also rich enough to encode the simplices, though not uniquely. We hope that its presheaves might form an interesting basis for a higher-dimensional type theory.

Abstract

A geometric n-dimensional cube has 2^n vertices as its extremal points, which are in bijection with the binary strings of length n, their *codes*. Two vertices bound an edge just in case their codes differ in exactly one index. We orient an edge so that the vertex whose code has 0 in this index is its domain, and that with 1 its codomain. These vertices and edges constitute a directed graph. Taking the preorder reflection of the free category on this graph gives us our notion of an n-dimensional *ordered cube*, $[n]$. So an ordered cube of any dimension is completely determined by its vertex set together with the "connectivity" of that set as determined by the directed edges.

For vertices $x, y : [n]$, we have $x \leq y$ just in case the code for x has 0 and that of y has 1 in all indices where they differ. Each $[n]$ forms a bounded distributive lattice, where $x \wedge y$ is the vertex whose code has 0 in the indices where those of x and y differ and their consensus otherwise; and dually for $x \vee y$.

A map of ordered cubes, $f : [m] \to [n]$, is a function from the 2^m vertices of $[m]$ to the 2^n vertices of $[n]$ that preserves the order relation on vertices. In other words, it is simply a map in $\textsc{PreOrd}\,([m] \to [n])$. So a map of ordered cubes is completely determined by its action on vertices. The *ordered cube category*, $\vec{\square}$, is the full subcategory of $\textsc{PreOrd}$ consisting of ordered cubes and their maps.

The ordered cube maps are generated by the dimension-raising maps $\vec{\square}\,([n] \to [n+1])$, the injective order-preserving functions on vertices, and the dimension-lowering maps $\vec{\square}\,([n+1] \to [n])$, the surjective order-preserving functions on vertices. Using these, we can define some familiar cube maps:

faces For index $0 \le i \le n$ and n-cube vertex $v : [n]$, inserting bit $b \in \{0,1\}$ into the code of v at index i (thus shifting all the subsequent bits) yields a vertex $v[i{\mapsto}b] : [n+1]$ which is in the b-face in dimension i of the $(n+1)$-cube.

degeneracies For index $0 \le i < n$ and n-cube vertex $v : [n]$, deleting the bit in the code of v at index i (thus shifting all the subsequent bits) yields a vertex $v(\hat{i}) : [n-1]$. This operation identifies vertices whose codes differed in only the deleted bit, preserves edges between vertices whose codes differed in only another bit, and inserts an edge between vertices whose codes differed in two bits, one of which was the deleted bit.

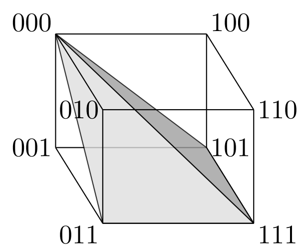
diagonals For bit $b \in \{0,1\}$, the map $b \longmapsto b^n : [1] \to [n]$ gives the main diagonal of an n-cube.

connections The map $\bigvee : [n] \to [1]$, which sends each vertex to the one represented by the supremum of the bits in its code implements the domain connection and the map $\bigwedge : [n] \to [1]$, which sends each vertex to the one represented by the infimum of the bits in its code implements the codomain connection.

The ordered cubes are cartesian with $[m] \times [n] \cong [m+n]$, with vertices coded by the concatenation of an $[m]$-vertex code and an $[n]$-vertex code.

The ordered cubes have lots of maps, which is both a blessing and a curse. One disadvantage is that the space required to specify a map $f : [m] \to [n]$ is exponential in m. However, in exchange for this we get a wealth of novel cubes, for example, the "bent" square:

$$\begin{array}{ccc} [2] & \xrightarrow{\beta} & [3] \\ 00 & \longmapsto & 000 \\ 01 & \longmapsto & 011 \\ 10 & \longmapsto & 101 \\ 11 & \longmapsto & 111 \end{array}$$



This map has as retraction the degeneracy that deletes the last index, $\hat{2} : [3] \to [2]$. But composing it with each of the other two possible degeneracies, $\hat{0}, \hat{1} : [3] \to [2]$ yields an ordered triangle. Indeed, the ordered cubes are expressive enough to encode the (ordered) simplices, though not uniquely.

At the workshop I will present some work in progress on the ordered cubes and their presheaves.