Cohesive Covering Theory (Extended Abstract)[1]

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A *monadic* modality in Homotopy Type Theory is a lot like an idempotent monad in category theory. The n-truncations and double negation are examples from plain Homotopy Type Theory. Viewing Homotopy Type Theory as an internal language of (∞,1)-toposes, which is crucial for the following, special features of a particular topos may be introduced into the type theory as a modality. Important parts of the coverings theory for topological spaces hold for a general abstract modality. In the following, this will be explained for a special type theory which admits recovery of the classic topological situation.

In [Shu15] Mike Shulman introduces Real Cohesive Homotopy Type Theory, as a candidate for an internal language of some of the (∞,1)-toposes called *cohesive*, a higher analog of Lawvere's axiomatic cohesion [Law07] developed by Urs Schreiber [Sch13].

In Real Cohesive Homotopy Type Theory, which will just be called Real Cohesion in the following, some well behaved topological spaces, like, for example, topological manifolds are supposed to be included in the theory. It is important to note, that the types corresponding to these topological spaces are 0-types in Real Cohesion. This can lead to confusion with the common explanation for the Identity types in Homotopy Type Theory, as paths in a space and care has to be taken to separate concept of equality and topological paths, i.e. maps $\gamma : \mathbb{R} \to X$ from the 0-type $\mathbb{R}$ representing the real line with the euclidean topology. Let $\mathbb{S}^n$ denote the topological sphere and S^n the higher inductive type introduced in [Uni13].

For the present work, the shape modality "$\int$" from Real Cohesion is of special interest. It maps topological spaces to their *homotopy type*, so for example $\int \mathbb{S}^1 = S^1$ and $\int \mathbb{R} = 1$. In a 1-topos cohesive over Set, the functor $\Delta \circ \pi_0$ maps a sheaf to the sheaf constantly its set of connected components. The $\int$ is a higher analog of this functor that extracts homotopical information on all h-levels, not just level 0. So if X represents a topological space with a point $*{:}X$, then $\int X$ is also pointed and the n-th homotopy group of X as a topological space could be retrieved from its shape as $\pi_n(X) :\equiv \|\Omega^n \int X\|_0$.

Like $\Delta \circ \pi_0$ reflects into the subcategory of constant sheaves, $\int$ reflects into the *discrete* types. Note that "discrete" refers to the topological structure of a type, not to a property of the ∞-groupoid structure given by its identity type.

As all modalities, $\int$ comes with a unit-map $\sigma_X : X \to \int X$, for any type X. For any two points $x, y : X$ that are joined by a topological path, the images $\sigma_X(x)$ and $\sigma_X(y)$ are equal in $\int X$.

Contribution. From this point on, let us assume that each type supposed to represent a topological space comes with a point "$*$" and for $* : X$ let us abbreviate $\sigma_X(*)$ with $*$.

For many modalities, the fibers of their units are interesting. For $\mathbb{S}^1$, this fiber

$$\sum_{x:\mathbb{S}^1} \sigma_{\mathbb{S}^1}(x) = *$$

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– or more precisely its projection to $\mathbb{S}^1$ – turns out to be the universal cover of $\mathbb{S}^1$. But this works only for spaces with trivial higher homotopy groups. For the construction of the universal cover of an arbitrary type, this has to be adjusted:

$$\widetilde{X} :\equiv \sum_{x\,:\,X} \|\sigma_X(x) = *\|_0.$$

Note that this amounts to replacing the modality $\int$ with the modality $\int_1 :\equiv \|\_\|_1 \circ \int$. To justify calling $\widetilde{X}$ the universal cover of X, we will define covering spaces relative to a modality and show a universal property.

For a modality $\bigcirc$ with unit η, we call a map $f : Y \to X$ a $\bigcirc$-cover, if the naturality square

$$\begin{tikzcd} Y \arrow[r, "\eta_Y"] \arrow[d, "f"'] \arrow[dr, phantom, "\text{(pb)}"] & \bigcirc Y \arrow[d, "\bigcirc f"] \\ X \arrow[r, "\eta_X"'] & \bigcirc X \end{tikzcd}$$

is a pullback square. For $\bigcirc \equiv \int_1$, covering spaces in topology have the universal property of this pullback for cones with a topological space as tip.

For a $\int_1$-cover $f : Y \to X$ there is always the trivial map $t : \widetilde{X} \to \int_1 Y$. Together with a $g : * =_{\int_1 X} *$ this yields a cone for the universal property of the defining pullback of f and therefore a unique map $\widetilde{X} \to Y$.

Similarly, the construction of covering spaces corresponding to subgroups of $\pi_1(X)$ can be done without using anything beyond the properties of a general modality: Any subgroup $H \subseteq \pi_1(X)$ can be represented by an action of $\pi_1(X)$ on a discrete 0-type [1] and therefore a map $BH \to \int_1 X$, with discrete BH. The pullback of this map is a $\int_1$-cover with the correct fiber type. A calculation on the level of abstract modalities shows, that this gives the usual one-to-one correspondence. In its most natural abstract form this correspondence of maps $M \to \int_1 X$ for modal M and $\int_1$-covers includes maps with discrete 1-types as fibers.

For the modality $\int$, this correspondence relates $\int$-covers with maps $M \to \int X$ for discrete M. Since the latter are ∞-actions of $\int X$ on discrete types, this seems to be a very natural generalization. The universal cover construction for $\int$ yields a "$\int$-universal cover" which can have both non-discrete topological structure and non-propositional identity types. This occurs whenever $\int X$ is not 1-truncated. For example, if we assume a type $\mathbb{CP}^\infty$ representing the appropriate topological space, the $\int$-universal cover will be a 1-type over $\mathbb{CP}^\infty$ with identity types merely S^1. The $\int$-universal cover $\widetilde{X}$ of a space has always a contractible shape, i.e. $\int X \simeq 1$.

Similar generalizations of the classical topological correspondence are known on the classical side for cohesive ∞-stacks [Sch13, Section 5.2.7] and topological 1-stacks [Noo05]. The author sees one advantage in the clarity of the type theoretic proofs, since the correspondence and some related remarks are all straight forward to prove using one lemma, which is so far the only formalized part of this work. The statement of this lemma is a generalization of the fact that modalities preserve products: For any dependent type $B : \bigcirc A \to \mathcal{U}$, we have $\bigcirc \sum_{x:A} B(\eta_A(x)) \simeq \sum_{x:\bigcirc A} \bigcirc B(x)$.

[1] This means we use the *homotopical* covering theory of [Hou17, Section 3.1] and [BvR18, Section 7.1]

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