Minimal models for monomial algebras*

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Abstract

Using combinatorics of chains going back to works of Anick, Green, Happel and Zacharia, we give, for any monomial algebra A, an explicit description of its minimal model. This also provides us with formulas for a canonical A_∞-structure on the Ext-algebra of the trivial A-module.

Introduction

Understanding A_∞-structures associated to differential graded associative (dga, for short) algebras is central to understanding in turn, the homotopy category of the category Alg of dga algebras. More precisely, one can, in principle, compute in the homotopy category of Alg by considering the category of quasi-free dga algebras or, equivalently, A_∞-coalgebras, modulo the usual relation of homotopy between morphisms in Alg: the quasi-free dga algebras are precisely the cofibrant objects of Alg, where the weak equivalences are the quasi-isomorphisms and the fibrations are the degree-wise epimorphisms; see [13, Théorème 1.3.1.1] and [12, 16].

In particular, we may use A_∞-coalgebras to understand usual (non-dg) associative algebras. For any augmented algebra A over a field $\Bbbk$, one can produce from the dga coalgebra $B(A)$, the bar complex of A, the equivalence class of minimal A_∞-coalgebra structures on $\operatorname{Tor}_A(\Bbbk,\Bbbk)$. Among other things, such structures determine A up to isomorphism, may be used to compute its Hochschild cohomology, and obtain the minimal model of A; see [12, 13]. The explicit computation of such higher structures is therefore of interest. The machinery of Gröbner bases and homological perturbation theory suggest that a possible first step towards solving this problem is to first obtain an answer for monomial algebras. In this paper we provide a complete description of a minimal A_∞-coalgebra structure on $\operatorname{Tor}_A(\Bbbk,\Bbbk)$

*The full version of this extended abstract can be found in math.KT:1804.01435 for a monomial algebra A in terms of the combinatorics of its Anick chains. Equivalently, we completely describe a minimal model of A as the cobar construction $\Omega_\infty \operatorname{Tor}_A$. The results extend without modification to describe minimal models of monomial quiver algebras in terms of the combinatorics of their chains; see [7].

Concretely, let γ be a basis element of $\operatorname{Tor}_A^{r+1}(\Bbbk,\Bbbk)$, represented by an Anick chain of length $r \in \mathbb{N}$ and let us take $n \in \mathbb{N}_{\geqslant 2}$. An *$n$-decomposition* of γ is a tuple $(\gamma_1,\dots,\gamma_n)$ of chains with lengths satisfying $r_1+\cdots+r_n = r-1$ and whose concatenation, in this order, is γ. Our result is the following.

Theorem. *For each monomial algebra A there is a minimal model $\mathscr{M} \longrightarrow A$ where $\mathscr{M} = \Omega_\infty \operatorname{Tor}_A(\Bbbk,\Bbbk)$ is the ∞-cobar construction on $\operatorname{Tor}_A(\Bbbk,\Bbbk)$. The differential b is such that for a chain $\gamma \in \operatorname{Tor}_A(\Bbbk,\Bbbk)$,*

$$b(s^{-1}\gamma) = \sum_{n\geqslant 2} (-1)^{\binom{n+1}{2}+|s^{-1}\gamma_1|} s^{-1}\gamma_1 \otimes \cdots \otimes s^{-1}\gamma_n,$$

where the sum ranges through all possible decompositions of γ.

This recovers, in particular, the results in [8] describing cup products in Ext_A for a monomial quiver algebra A using a multiplicative basis of chains, and the results in [9] describing the A_∞-algebra structure of Ext_A for monomial algebras which are p-Koszul.

1 Recollections

1.1 Algebraic discrete Morse theory

(1.1.1) Let C be a non-negative complex of free $\Bbbk$-modules. Fix a basis $X = X_0 \cup X_1 \cup \cdots$ of homogeneous elements of C, so that for each $t \in \mathbb{N}_0$, the set X_t is a basis of C_t. Given $c \in X$ we introduce the notation $dc = \sum_{c'\in X}[c:c']c'$ where $[c:c'] \in \Bbbk$. Let $G = G(C,X)$ be the directed weighted graph with vertices the set X and with an edge $c \to c'$ if c' appears in dc with non-zero coefficient $[c:c']$ which is, in that case, the weight of $c \to c'$. A finite subset M of edges of G is a *Morse matching* if it satisfies the following Morse conditions:

- **M1.** Each vertex of G is in at most one edge of M.
- **M2.** The weights of edges of M are invertible.
- **M3.** The graph G_M obtained by inverting the edges of M in G has no directed cycles.

If $c' \to c$ is a edge in G_M with $c \to c' \in M$, we set its weight to be $-[c:c']^{-1}$. In our situation the coefficients $[c:c']$ will be either 1 or -1, which means **M2** is always satisfied. We write

X^M for the collection of vertices not appearing in M, which we call *critical*. Assign a path the product of the weights of the edges it contains. Finally, write $\Gamma(c,c')$ for the sum of all the weights of paths from c to c' in G_M. The main theorem of [11] is the following.

Theorem 1.1. *The complex C^M is homotopy equivalent to C, and the maps $f : C \longrightarrow C^M$, $g : C^M \longrightarrow C$ given on basis elements by $f(c) = \sum_{c'\in X_t^M} \Gamma(c,c')c'$, $g(c) = \sum_{c'\in X_t} \Gamma(c,c')c'$ for $c \in X_t$, respectively $c \in X_t^M$, are inverse homotopy equivalences. In fact, $fg = 1$ and $gf - 1 = dh + hd$ where for a basis element $c \in X_t$, $h(c) = \sum_{c'\in X_{t+1}} \Gamma(c,c')c'$.*

1.2 Anick's resolution via Algebraic discrete Morse theory

(1.2.1) We now describe the critical vertices B^M. Let $m_1,\dots,m_{l-1}$ be minimal monomial generators of the ideal of leading monomials of I, such that for each $j \in \{1,\dots,l-1\}$ we have $m_j = u_j v_j u_{j+1}$ where u_1 is a variable. We call the term $[u_1|v_1u_2|v_2u_3|\cdots|v_{l-1}u_l]$ *fully attached* if for all $j \in \{1,\dots,l-2\}$ and each prefix u of $v_{j+1}u_{j+2}$ the monomial $v_ju_{j+1}u$ is normal. We denote by B_j the set of fully attached terms of degree $j \geqslant 2$ and let B_1 consist of the variables. Let us say that a pair (u,v) of monomials has zero product *minimally* if $uv = 0$ but uv' is nonzero for any left divisor v' of v. For the proofs of the following two lemmas we refer the reader to [11].

Lemma 1.2. *Assume that A is a monomial algebra. Let $j \in \mathbb{N}$. There is a Morse matching $M = M_1 \cup M_2 \cup \cdots$ on the bar resolution $B(A,A)$ of A for which elements of M_j consist of those edges of the form $[x_i|u_1|\cdots|u_{j-1}|u_j|\cdots] \to [x_i|u_1|\cdots|u_{j-1}u_j|\cdots]$ such that $x_iu_1 = u_1u_2 = \cdots = u_{j-2}u_{j-1} = 0$ minimally and $u_{j-1}u_j \neq 0$.*

Lemma 1.3. *The fully attached tuples are exactly the critical vertices, and $B(A,A)^M$ is the Anick resolution of A. In case A is monomial, the critical vertices are the variables $[x_1],\dots,[x_n]$ along with those terms $[x_i|u_1|\cdots|u_r]$ where if we set $x_i = u_0$, $u_ju_{j+1} = 0$ minimally for $j \in \{0,\dots,r-1\}$.*

1.3 Homotopy transfer theorem and A_∞-coalgebras

(1.3.1) Let C be a dga coalgebra, and assume that V is a complex of $\Bbbk$-modules which is a *deformation retract of* C given by maps $i : V \to C$, $p : C \to V$ and $h : C \to C$. By this we mean that $pi = 1$ and $ip - 1 = dh + hd$. We assume that such data satisfies the *side conditions*, that is, all three maps h^2, hi and ph are zero. The following result of [15] shows how to transfer on V a structure of A_∞-coalgebra from the dga coalgebra structure of C and, further, how to produce from the homotopy data another homotopy data of A_∞-coalgebras.

Theorem 1.4 (Homotopy Transfer Theorem)**.** *Let (C,Δ_2') be a dga coalgebra and consider a homotopy retract as above. There exists an A_∞-coalgebra structure on V and a homotopy retract data from $\Omega_\infty C$ to $\Omega_\infty V$. The A_∞-coalgebra structure on V is given by $\Delta_1 = d_V$ and, for $n \in \mathbb{N}_{\geqslant 2}$, by $\Delta_n = p^{\otimes n}\Delta_n' i$, where for $n \in \mathbb{N}_{\geqslant 3}$ the arrows $\Delta_n' : C \longrightarrow C^{\otimes n}$ are defined by*

$$\Delta_n' = \sum_{\substack{s+t=n \\ s,t>0}} (-1)^{s(t+1)}(\Delta_s' h \otimes \Delta_t' h)\Delta_2',$$

with the convention that $\Delta_1' h = 1$. ◀

(1.3.2) If T is a planar binary tree of n leaves, write Δ_T for the cooperation of arity n obtained by decorating the leaves of T with p, the root of T with i, the inner vertices with Δ_2' and the inner edges with h. We define an integer $\vartheta(T)$ as follows. For each vertex v of T, let r_1 be the number of paths from a leaf of T to the root that pass through the first input of v, and let r_2 be the number of those that pass through the second. Set $\vartheta_T(v) = r_1(r_2+1)$ and $\vartheta(T) = \sum_{v\in T}\vartheta_T(v)$. Let us write Δ_T for the operator obtained by decorating the leaves of T by p, the root of T by i, the inner vertices by Δ_2' and the inner edges by h. We then have the following result. We then have the following result of [15].

Theorem 1.5. *Let $n \in \mathbb{N}$. Then Δ_n is given by the sum $\sum_T (-1)^{\vartheta(T)}\Delta_T$ as T ranges through all planar binary trees with n leaves.* ◀

2 Coproducts act by the right comb

2.1 A_∞-structure on Tor

(2.1.1) In this subsection we will prove that for each $n \in \mathbb{N}_0$, following the description of the higher coproducts of Tor_A given by Theorem 1.5 by taking $C = BA$, $V = \mathrm{Tor}_A$ and the contraction afforded by [11], the only tree that contributes to the computation of Δ_n is the right comb.

(2.1.2) Suppose that $\gamma = [u_0|\cdots|u_r]$ is attached but is not a chain. Then there is a largest i_1 such that $u_i = u_i' u_i''$ and such that $\eta^1 = [u_0|\cdots|u_i']$ is a chain. It may happen that $i = 0$, in which case u_0' is simply the first variable in u_0, as it does for $[t^2|t]$. We define

$$\gamma^1 = (-1)^{i_1+1}[\eta^1|u_{i_1}{}''|u_{i_1+1}|\cdots|u_r], \quad \Gamma^1 = [\eta^1|u_{i_1}{}''u_{i_1+1}|\cdots|u_r].$$

If Γ^1 is a chain or zero, stop. Else, there is some largest $i_2 > i_1$ such that, keeping in with the notation above, $\eta = [u_0|\cdots|u'_{i_1}|\cdots|u'_{i_2}]$ is a chain. In which case, set

$$\gamma^2 = (-1)^{i_2+1}[\eta^2|u''_{i_2}|\cdots|u_r], \quad \Gamma^2 = [\eta^2|u''_{i_2}u_{i_2+1}|\cdots|u_r].$$

Continuing in this way, we obtain terms $\gamma = \Gamma^0, \cdots, \Gamma^n$ and $\gamma^1, \cdots, \gamma^n$, where Γ^n is either zero or a chain.

Lemma 2.1. *With the notation above, we have that* $h(\gamma) = \sum_{i=1}^n \gamma^i$, $p(\gamma) = \Gamma^n$. ◀

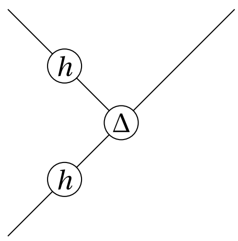
(2.1.3) In the language of the Morse graph associated to the matching M of BA given by Lemma 1.2, whose critical set of vertices is Tor_A, we have the following corollary.

Corollary 2.2. *Let c be a vertex in G^M of degree t that is not critical. There is a unique element c' of degree $t+1$ and a unique element c'' of degree t, which is either zero or critical, a unique path in G^M from c to c' and, if c'' is nonzero, a unique edge from c' to c''. Thus, the coefficients in the homotopy of Theorem 1.1 are all 1 or -1 and $p(c)$ coincides with c'' up to a sign.* ◀

(2.1.4) The following exchange rule between Δ'_2 and h shows that most of the trees in the definition of the higher coproducts in Tor_A do not contribute to their computations.

Lemma 2.3. *Suppose that γ is attached. Then $\Delta'_2(h(\gamma)) = (h\otimes 1)\Delta'_2(\gamma)$ modulo $\mathrm{Tor}_A \otimes B(A)$. It follows that have $(h\otimes 1)\Delta'_2 h = 0$ on attached terms. It follows that if $n \in \mathbb{N}_{\geqslant 3}$ and if $\gamma \in \mathrm{Tor}_A$ is an element represented by an Anick chain, then only tree that contributes to $\Delta'_n(\gamma)$, and hence to $\Delta_n(\gamma)$, is the right comb.*

Proof. Let us explain how the second claim follows. The fact that h vanishes on Tor_A means that, at the root, the left edge must be a leaf. Knowing this, the exchange rule means that if T is planar and contains any subtree of the form



which corresponds to $(h\otimes 1)(\Delta'_2 h)$, the operator Δ_T will vanish identically. This means that T can only have leaves growing to the left, and hence T can only be the right comb. ◀

3 The description of the minimal model

3.1 Combinatorics of chains and tails

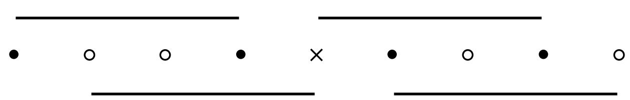
We now aim to give a more refined description of the terms appearing in a higher coproduct of a fixed chain γ, as stated in the following theorem.

Theorem 3.1. *Let γ be a chain and $n \in \mathbb{N}_{\geqslant 2}$. The terms that appear in $\Delta_n(\gamma)$ are exactly those of the form $\gamma_1 \otimes \cdots \otimes \gamma_n$ with $(\gamma_1, \ldots, \gamma_n)$ a decomposition of γ. Moreover, the coefficient of $\gamma_1 \otimes \cdots \otimes \gamma_n$ is $(-1)^N$ where $N = \binom{n+1}{2} + r_1 + \sum_{i=1}^{n-1}(n-i)(r_i+1)$.*

(3.1.1) Suppose that $\gamma = x_{i_1} \cdots x_{i_s}$ is an Anick chain, with associated interlaced sequences $\{(a_j),(b_j)\}$, that is

$$1 = a_1 < a_2 \leqslant b_1 < a_3 \leqslant b_2 < \cdots < a_r \leqslant b_{r-1} < b_r = s,$$

and each $x_{i_{a_j}} \cdots x_{i_{b_j}}$ is a relation. We will say a variable x_{i_j} is an overlapping variable if $s \in [a_{j+1}, b_j)$, and we will say that a bar is *inserted at x_{i_j}* if it is inserted immediately after it. It may happen that $a_{j+1} = b_j$, in which case we agree that $x_{i_{a_{j+1}}}$ is both overlapping and non- overlapping. This always happens, for example, if A is quadratic. A bar term obtained from γ is *regular* if it is obtained by inserting bars at non-overlapping variables, and it is *coregular* if it is obtained by inserting bars at overlapping variables. The following figure illustrates our definitions for the 4-chain $[t|t^3|t|t^3|t]$ in $\Bbbk\langle t|t^4\rangle$, where white circles represent overlapping variables, black ones represent non-overlapping variables, the cross represents the only variable that is both overlapping and non-overlapping, and bars mark the obstructions that constitute the chain.



Lemma 3.2. *Let γ be a monomial which is an r-chain. Any regular bar term obtained by inserting*

(1) exactly r bars into γ is either attached and nonzero or is zero,
(2) less than r bars into γ is zero, and
(3) more than r bars into γ is not attached and nonzero or is zero.

Analogous statements hold for coregular bar terms. ◀

(3.1.2) We now note that the homotopy h, which introduces and shifts bars in bar terms, produces bar terms whose subchains, starting from the left, have bars introduced regularly.

Lemma 3.3. *If γ is an element of $\operatorname{Tor}_A^{r+1}$ corresponding to an r-chain, it has its r bars inserted regularly. In particular, if γ is an attached term, and if γ^a is a nonzero summand in $h(\gamma)$, following the notation of Lemma 2.1, then for $j \leqslant i_a$, the j-chain $(\gamma^a)_{(j+1)}$ has its j bars inserted regularly.*

(3.1.3) Let us now introduce the definitions that will be central to our proof of Theorems 3.1 and its equivalent formulation in terms of minimal models already stated in the Introduction. Let γ be an r-chain and $j \in \mathbb{N}$. We will say a bar term Γ is a *j-tail of γ* if there is a term of the form $\gamma_1 \otimes \cdots \otimes \gamma_j \otimes \Gamma$ in $\Delta'_{j+1}(\gamma)$ appearing with nonzero coefficient, where the first j tensors are chains, and, moreover, Γ is a concatenation of at least two chains $\gamma_{j+1}, \ldots, \gamma_n$, in this order. Moreover, if for $i \in [n]$ we have that γ_i is an r_i chain, we require that $r_1 + \cdots + r_n = r - 1$. The *length* of Γ is $n - j$. Let us call the n-tuple $(\gamma_1, \ldots, \gamma_n)$ a *decomposition* of γ. Remark that there is the notion of "tail" of a chain given in [1], but that this is not a special case of our definition.

(3.1.4) The following lemma is central to our result.

Lemma 3.4. *Fix $j \in \mathbb{N}$ and suppose that $\gamma = [u_0|u_1|\cdots|u_r]$ is an r-chain, and that Γ is a j-tail of γ, with first chain γ_{j+1}. Then there exists $i \in \{1, \ldots, r\}$ and a decomposition $u_i = u'_i u''_i$ such that $u''_i \neq 1$, $u''_i u_{i+1} = 0$ minimally and $\Gamma = [u''_i|\cdots|u_r]$. Moreover:*

(1) This decomposition is nontrivial whenever $j > 1$
(2) The tail Γ contains exactly $r_{j+1} + \cdots + r_n$ bars.
(3) There is a unique $(j-1)$-tail Γ' and a unique term in $\Delta'_2 h(\Gamma')$ of the form $\gamma_j \otimes \Gamma$ that gives rise to Γ, and it appears with a sign as a coefficient.

Remark that the condition $u''_i u_{i+1} = 0$ is vacuous if Γ happens to have no bars.

(3.1.5) The following proposition is the main result about tails and chains we were after, and follows at once from Lemma 3.4. Let γ be an r-chain and $n \in \mathbb{N}_{\geqslant 2}$. An *$n$-decomposition of γ* is a tuple $(\gamma_1, \ldots, \gamma_n)$ of chains such that the underlying monomial of the concatenation $\gamma_1 \cdots \gamma_n$ is γ and such that $r_1 + \cdots + r_n = r - 1$ where r_i is the lenght of γ_i.

Proposition 3.5. *Let γ be a chain, $n \in \mathbb{N}_{\geqslant 2}$ and let $(\gamma_1, \ldots, \gamma_n)$ be a decomposition of γ. For each $j \in [n-1]$ there is a unique j-tail Γ of γ with underlying monomial $\gamma_{j+1} \cdots \gamma_n$ and a unique term $\gamma_1 \otimes \cdots \otimes \gamma_j \otimes \Gamma$ in $\Delta'_{j+1}(\gamma)$, and it appears with coefficient 1 or -1.* ◀

3.2 Main theorem

(3.2.1) We now recall the promised description of the minimal model of a monomial algebra A, which follows immediately from the last proposition and the book-keeping of signs made in Theorem 3.1. Before doing so, let us remark that all of the work was done for monomial algebras for readability, but that our work is equally valid for monomial quiver algebras.

(3.2.2) Fix a quiver $Q = (Q_0, Q_1, s, t)$ and a set R of paths in Q of length at least two, none of which is a divisor of another. We call $A = \Bbbk Q/(R)$ a *monomial quiver algebra*. Let us write $\underline{\Bbbk}$ for the semi-simple $\Bbbk$-algebra $\Bbbk Q_0$, so that there is an augmentation $A \longrightarrow \underline{\Bbbk}$. We set $\mathrm{Tor}_A = \mathrm{Tor}_A(\underline{\Bbbk}, \underline{\Bbbk})$, and write $B(A)$ for the bar complex of A, where unadorned $\otimes$ are now taken over $\underline{\Bbbk}$. Thus, a generic basis element of $B(A)$ in degree $n \in \mathbb{N}$ is of the form $[a_1|\cdots|a_n]$ where $t(a_i) = s(a_{i+1})$ for each $i \in \{1, \ldots, n-1\}$.

(3.2.3) The notion of decompositions of a chain carry through to this setting, as well as the technical work of Sections 2 and 3. As an end result we obtain, mutatis mutandis, the following description of a minimal model for quiver monomial algebras. Naturally, we have a dual result for the Yoneda algebra $\mathrm{Ext}_A(\underline{\Bbbk}, \underline{\Bbbk})$ of A, which we also record.

Theorem 3.6. *Let $A = \Bbbk Q/(R)$ be a quiver monomial algebra. There is a minimal model $\mathscr{M} \longrightarrow A$ where $\mathscr{M} = \Omega_\infty \mathrm{Tor}_A(\Bbbk, \Bbbk)$ is the ∞-cobar construction on $\mathrm{Tor}_A(\Bbbk, \Bbbk)$. The differential b is such that for a chain $\gamma \in \mathrm{Tor}_A(\Bbbk, \Bbbk)$,*

$$b(s^{-1}\gamma) = \sum_{n \geqslant 2} (-1)^{\binom{n+1}{2} + |s^{-1}\gamma_1|} s^{-1}\gamma_1 \otimes \cdots \otimes s^{-1}\gamma_n,$$

where the sum ranges through all possible decompositions of γ. ◀

Theorem 3.7. *There is a canonical A_∞-algebra structure on Ext_A given as follows. If $n \in \mathbb{N}_{\geqslant 2}$ and if $\gamma_1^\vee, \ldots, \gamma_n^\vee$ are chains in Ext_A of lengths $r_1, \ldots, r_n$, respectively, then $\mu_n(\gamma_1^\vee \otimes \cdots \otimes \gamma_n^\vee) = (-1)^M \gamma^\vee$ if the concatenation $\gamma_1 \cdots \gamma_n$ is a chain of length $r = r_1 + \cdots + r_n + 1$ where M is the integer given by the sum $M = \binom{n+1}{2} - 1 + \sum_{i<j} r_i(r_j + 1) + r_1 + r$. Otherwise, this higher product is zero.* ◀

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