Parameterized complexity of games with monotonically ordered ω-regular objectives

Véronique Bruyère[1], Quentin Hautem[1⋆], and Jean-François Raskin[2]

[1]Département d'informatique, Université de Mons (UMONS), Mons, Belgium
[2] Département d'informatique, Université libre de Bruxelles (U.L.B.), Brussels, Belgium

Abstract. In recent years, two-player zero-sum games with multiple objectives have received a lot of interest as a model for the synthesis of complex reactive systems. In this framework, Player 1 wins if he can ensure that all objectives are satisfied against any behavior of Player 2. When this is not possible to satisfy all the objectives at once, an alternative is to use some preorder on the objectives according to which subset of objectives Player 1 wants to satisfy. For example, it is often natural to provide more significance to one objective over another, a situation that can be modelled with lexicographically ordered objectives for instance. Inspired by recent work on concurrent games with multiple ω-regular objectives by Bouyer et al., we investigate in detail turned-based games with monotonically ordered and ω-regular objectives. We study the threshold problem which asks whether Player 1 can ensure a payoff greater than or equal to a given threshold w.r.t. a given monotonic preorder. As the number of objectives is usually much smaller than the size of the game graph, we provide a parametric complexity analysis and we show that our threshold problem is in FPT for all monotonic preorders and all classical types of ω-regular objectives. We also provide polynomial time algorithms for Büchi, coBüchi and explicit Muller objectives for a large subclass of monotonic preorders that includes among others the lexicographic preorder. In the particular case of lexicographic preorder, we also study the complexity of computing the values and the memory requirements of optimal strategies.

1 Introduction

Two-player zero-sum games played on directed graphs form an adequate framework for the *synthesis of reactive systems* facing an uncontrollable environment [12]. To model properties to be enforced by the reactive system within its environment, games with Boolean objectives and games with quantitative objectives have been studied, for example games with ω-regular objectives [10] and mean-payoff games [14].

Recently, games with *multiple* objectives have received a lot of attention since in practice, a system must usually satisfy several properties. In this framework, the system wins if it can ensure that *all* objectives are satisfied no matter how the environment behaves. For instance, generalized parity games are studied in [8], multi-mean-payoff games in [13], and multidimensional games with heterogeneous ω-regular objectives in [4].

When multiple objectives are conflicting or if there does not exist a strategy that can enforce all of them at the same time, it is natural to consider trade-offs. A general framework for defining trade-offs between n (Boolean) objectives $\Omega_1, \dots, \Omega_n$ consists in assigning to each infinite path π of the game a payoff $v \in \{0,1\}^n$ such that $v(i) = 1$ iff π satisfies Ω_i, and then to equip $\{0,1\}^n$ with a preorder $\precsim$ to define a preference between pairs of payoffs: $v \precsim v'$ whenever payoff v' is preferred to payoff v. Because the ideal situation would be to satisfy *all* the objectives together, it is natural to assume that the preorder $\precsim$ has the following *monotonicity* property: if v' is such that whenever $v(i) = 1$ then $v'(i) = 1$, then it should be the case that v' is preferred to v. Classical examples are the subset preorder, the maximize preorder and the lexicographic preorder (see for instance [3]).

For this talk, we consider the following threshold problem: given a game graph G, a set of *ω-regular objectives*[1] $\Omega_1, \dots, \Omega_n$, a monotonic preorder $\precsim$ on the set $\{0,1\}^n$ of payoffs, and a threshold μ, decide

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[1] We cover all classical ω-regular objectives: reachability, safety, Büchi, co-Büchi, parity, Rabin, Streett, explicit Muller, or Muller.

whether Player 1 has a strategy such that for all strategies of Player 2, the outcome of the game has payoff v greater than or equal to μ (for the specified preorder), i.e. $\mu \precsim v$. As the number n of objectives is typically much smaller than the size of the game graph G, it is natural to consider a parametric analysis of the complexity of the threshold problem in which the number of objectives and their size are considered to be fixed parameters of the problem. Our main results are as follows (full paper available on ArXiV [5]).

Results presented in this talk. First, we provide *fixed parameter tractable solutions* to the threshold problem for *all* monotonic preorders and for *all* classical types of ω-regular objectives. Our solutions rely on the following ingredients:

1. We show that solving the threshold problem is equivalent to solve a game with a single objective Ω that is a union of intersections of objectives taken among $\Omega_1, \ldots, \Omega_n$. This is possible by *embedding* the monotonic preorder $\precsim$ in the subset preorder and by translating the threshold μ in preorder $\precsim$ into an antichain of thresholds in the subset preorder. A threshold in the subset preorder is naturally associated with a conjunction of objectives, and an antichain of thresholds leads to a union of such conjunctions.
2. We provide a fixed parameter tractable algorithm to solve games with a single objective Ω as described previously for all types of ω-regular objectives $\Omega_1, \ldots, \Omega_n$, leading to a fixed parameter algorithm for our threshold problem. Those results build on the recent breakthrough of Calude et al. that provides a quasipolynomial time algorithm for parity games as well as their fixed parameter tractability [7], and on the fixed parameter tractability of games with an objective defined by a Boolean combination of Büchi objectives.

Second, we consider games with a preorder $\precsim$ having a *compact embedding*, with the main condition that the antichain of thresholds resulting from the embedding in the subset preorder is of *polynomial size*. The maximize preorder, the subset preorder, and the lexicographic preorder, given as examples above, all possess this property. For games with a compact embedding, we go beyond fixed parameter tractability as we are able to provide deterministic polynomial time solutions for Büchi, coBüchi, and explicit Muller objectives. Polynomial time solutions are not possible for the other types of ω-regular objectives as we show that the threshold problem for the lexicographic preorder with reachability, safety, parity, Rabin, Streett, and Muller objectives cannot be solved in polynomial time unless $\mathsf{P} = \mathsf{PSPACE}$. Finally, we present a full picture of the study of the lexicographic preorder for each studied objective. We give the exact complexity class of the threshold problem, show that we can obtain the values from the threshold problem (which thus yields a polynomial algorithm for Büchi, co-Büchi and Explicit Muller objectives, and an FPT algorithm for the other objectives) and provide tight memory requirements for the optimal and winning strategies.

Related work. In [3], Bouyer et al. investigate concurrent games with multiple objectives leading to payoffs in $\{0,1\}^n$ which are ordered using Boolean circuits. While their threshold problem is slightly more general than ours, their games being concurrent and their preorders being not necessarily monotonic, the algorithms that they provide are nondeterministic and guess witnesses whose size depends polynomially not only in the number of objectives but also in the size of the game graph. Their algorithms are sufficient to establish membership to PSPACE for all classical types of ω-regular objectives but they do not provide a basis for the parametric complexity analysis of the threshold problem. In stark contrast, we provide deterministic algorithms whose complexity only depends polynomially in the size of the game graph. Our new deterministic algorithms are thus instrumental to a finer complexity analysis that leads to fixed parameter tractability for all monotonic preorders and all ω-regular objectives. We also provide tighter lower-bounds for the important special case of lexicographic preorder, in particular for parity objectives.

The particular class of games with multiple Büchi objectives ordered with the maximize preorder has been considered in [1]. The interested reader will find in that paper clear practical motivations for considering multiple objectives and ordering them. The lexicographic ordering of objectives has also been considered in the context of quantitative games: lexicographic mean-payoff games in [2], some special cases of lexicographic quantitative games in [6,11], and lexicographically ordered energy objectives in [9].

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