Algorithm for Bernstein Polynomial Control Design

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Abstract:
This paper considers control synthesis for polynomial control systems. The developed method leans upon Lyapunov stability and Bernstein certificates of positivity. We strive to develop an algorithm that computes a polynomial control and a polynomial Lyaponov function in the simplicial Bernstein form. Subsequently, we reduce the control synthesis problem to a finite number of evaluations of a polynomial within Bernstein coefficient bounds representing controls and Lyapunov functions. As a consequence, the equilibrium is asymptotically stable with this control.

Keywords: Bernstein basis, certificates of positivity, control design, subdivision of simplex, stability.

1. INTRODUCTION

We use the Lyapunov stability of control systems to develop an algorithm that computes a pair (u, v) of a polynomial Lyapunov v and a polynomial control u in the Bernstein form. Computing of such a pair ensures asymptotic stability of the designed feedback system. Many researchers have focused on the topic of stability analysis for nonlinear dynamic systems, which continues to be a challenging problem Ackerman (1993); Nersesov and Haddad (2006). The range of polynomial functions over a simplex is bounded by the smallest and the largest Bernstein coefficients, enclosure bound. We investigate certificates of positivity of polynomials in the Bernstein basis with respect to degree elevation and with respect to the maximum diameter of subsimplices Leroy (2009); Hamadneh and Wisniewski (2017). Our strategy of Bernstein coefficient bounds is motivated in Ribard et al. (2016). However, we additionally develop an algorithm for shrinking of general coefficient bounds. We use the Bernstein coefficients to derive bounds for u and v. Bounding of polynomial functions provides general certificates of positivity over a given domain (simplex). Computing such bounds for the range (minimum and maximum) of values of a polynomial has received a good deal of attention in the past Lane and Riesenfeld (1980); Rokne (1977); Cargo and Shisha (1966); Garloff (1986); Smith (2012); Garloff and Hamadneh (2015). Peteres Nairn et al. (1999) first provided the convergence properties of the sequence of control polygons to the Bernstein-Bezier segment with respect to subdivision and degree elevation. The extension to the multivariate case over a simplex is given in Leroy (2009), Titi et al. (2015) and Hamadneh (2018). A method for solving systems within an n-dimensional simplex, which relies on the representation of polynomials in the barycentric Bernstein basis is given in Reuter et al. (2008). As a consequence, the stability of the designed feedback system has been translated to certificates of positivity. The Lie derivative by Lyapunov stability of control systems should be negative semidefinite in the neighborhood of the equilibrium. By subdivision at the equilibrium point, we show that the enclosure bound of polynomials over a union of subsimplices is contained in the enclosure bound over the whole simplex. This leads to local certificates of positivity.

Computing a Lyapunov function of polynomial vector fields has also attracted the interest of many researchers in the past Hou and Michel (2007); Ackerman (1993); Nersesov and Haddad (2006); Sloth and Wisniewski (2014). The approach we have chosen for computing a Lyapunov function for a polynomial system is to provide a certificate of positivity. Algorithms for using polynomial certificates of positivity for computing Lyapunov functions are given in Kamyar et al. (2014); Habets et al. (2006). The existence of the popular sum of squares certificate of positivity can be formulated as a semidefinite programming problem Parrilo (2003). In fact, the previous algorithms for computing (u, v) suffer from complexity issues, e.g., Hill and Moylan (1977); Riener et al. (2013). In this paper, we will use the certificates of positivity in the simplicial Bernstein basis for the general designing control systems for polynomial dynamic systems. By applying the barycentric coordinate of a simplex, we provide a local certificate of positivity for a Lyapunov function in the Bernstein basis. Subsequently, we provide a new technique for computing a pair (u, v) of control and Lyapunov functions of a finite Bernstein degree. Finally, we provide an algorithm for tightening the coefficient bounds that give an estimation of the control and Lyapunov functions.

The organization of our paper is as follows: In the next section, we briefly recall the Bernstein form and its basic properties. The main results are given in Section 3. Finally, Section 4 comprises conclusions.

2. BACKGROUND

Throughout this paper, we use polynomials in the Bernstein basis over simplices. Specifically, we apply the Bernstein basis for the stability of polynomial dynamic systems since the Bernstein basis is positive semidefinite over a simplex V, i.e., $B_\alpha^{(k)}(x) \geq 0,\ \forall x \in V$. For completeness of the exposition, we introduce some notation and essential background about the simplicial Bernstein basis. Throughout the paper, $V = [\sigma_0, \ldots, \sigma_n]$ will denote a non-degenerate simplex of $\mathbb{R}^n$; viz the points $\sigma_0, \ldots, \sigma_n$ are affinely independent. Let $\lambda_0, \ldots, \lambda_n$ be the associated barycentric coordinates of V, i.e., the linear polynomials of $\mathbb{R}[X] = \mathbb{R}[X_1, \ldots, X_n]$ such that $\sum_{i=0}^n \lambda_i(x) = 1$ and $\forall x \in \mathbb{R}^n,\ x = \lambda_0(x)\sigma_0 + \cdots + \lambda_n(x)\sigma_n$. The realization $|V|$ of the simplex V is the subset of $\mathbb{R}^n$ defined as the convex hull of the points $\sigma_0, ..., \sigma_n$. Without loss of generality, we can assume that V is the standard simplex $\Delta = [e_0, e_1, \ldots, e_n]$, where $(e_1, \ldots, e_n)$ denotes the canonical basis of $\mathbb{R}^n$, and $e_0 = (0, ..., 0)$ is the origin. This is not a restriction since any simplex V in $\mathbb{R}^n$ can be mapped affinely upon Δ. Specifically, if $x = (x_1, ..., x_n) \in \Delta$, then $(\lambda_0, ..., \lambda_n) = (1 - \sum_{i=1}^n x_i, x_1, ..., x_n)$.

We refer to the multi-index $\alpha = (\alpha_0, \ldots, \alpha_n) \in \mathbb{N}^{n+1}$ and $|\alpha| = \alpha_0 + \cdots + \alpha_n$. For $\hat{\beta} = (\beta_1, ..., \beta_n),\ \hat{\alpha} = (\alpha_1, ..., \alpha_n)$ with $\hat{\beta} \leq \hat{\alpha}$ (component-wise), we define

$$\binom{\hat{\alpha}}{\hat{\beta}} := \prod_{i=1}^n \binom{\alpha_i}{\beta_i}.$$

If k is a natural number such that $|\hat{\beta}| \leq k$, we use the notation $\binom{k}{\hat{\beta}} := \frac{k!}{\beta_1! \ldots \beta_n! (k - |\hat{\beta}|)!}$.

For $x \in \mathbb{R}^n$ its multi-powers are $x^{\hat{\beta}} := \prod_{i=1}^n x_i^{\beta_i}$. Let f be a polynomial (in the monomial form) function of degree l,

$$f(x) = \sum_{|\hat{\beta}| \leq l} a_{\hat{\beta}} x^{\hat{\beta}}, \tag{1}$$

f can be uniquely expresses for $l \leq k$ as

$$f(x) = \sum_{|\alpha| = k} b_\alpha(f, k, \Delta) B_\alpha^{(k)}(x), \tag{2}$$

where $b_\alpha(f, k, \Delta)$ are called the Bernstein coefficients of f of degree k with respect to Δ given as

$$b_\alpha(f, k, \Delta) = \sum_{\hat{\beta} \leq \hat{\alpha}} \frac{\binom{\hat{\alpha}}{\hat{\beta}}}{\binom{k}{\hat{\beta}}} a_{\hat{\beta}}. \tag{3}$$

The Bernstein polynomials of degree k with respect to Δ are the polynomials $(B_\alpha^{(k)})_{|\alpha|=k}$, where

$$B_\alpha^{(k)}(\lambda) = \binom{k}{\alpha} \lambda^\alpha. \tag{4}$$

The grid points of degree k associated to Δ are the points

$$\sigma_\alpha(k, \Delta) = \frac{\alpha_0 e_0 + \cdots + \alpha_n e_n}{k} \in \mathbb{R}^n\ (|\alpha| = k), \tag{5}$$

whereas, the control points associated to f are

$$(\sigma_\alpha(k, \Delta),\ b_\alpha(f, k, \Delta)) \in\ \mathbb{R}^{n+1}\ (|\alpha| = k). \tag{6}$$

The set of control points of f forms its control net of degree k.

Finally, the discrete polynomial over Δ is defined as $f(\sigma_\alpha(k, \Delta))$.

3. MAIN RESULTS

In this section, we devise an algorithm for control synthesis. Specifically, we will translate the control synthesis problem to finding a pair of control and Lyapunov functions such that certain bilinear inequalities hold. If there exists such (u, v), we will briefly say that there exists a stabilizing control. We suppose that all vector fields are polynomials defined on a union of simplices, $\Delta = W^{[1]} \cup ... \cup W^{[s]}$. To this end, we use the representation of all polynomials in the simplicial Bernstein basis. The considered affine control system is given by

$$\dot{x} = F_u(x) = p(x) + g(x)u(x), \tag{7}$$

where the vector field $F_u : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is defined by the drift $p : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and the control $u : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ with the input matrix function $g : \mathbb{R}^n \longrightarrow \mathbb{R}^{n \times m}$.

We follow the definition of asymptotic and stability, and recall the condition of Lyapunov stability.

Definition 1. Let x_0 be an equilibrium point for (7) and let $A \subseteq \mathbb{R}^n$ be a collection of simplices containing the interior point x_0. Let $v : A \longrightarrow \mathbb{R}$ be a continuously differentiable function such that $v(x_0) = 0$,

$$v(x) > 0,\ \forall x \in A \setminus \{x_0\},$$

$$\mathcal{L}_{F_u}(v)(x) = -\frac{\partial v}{\partial x}(x) F_u(x) > 0,\ \forall x \in A \setminus \{x_0\},$$

where $\mathcal{L}$ denotes the negative Lie derivative. Then v will be called a strict Lyapunov function for F_u.

Specifically, if there exist a polynomial Lyapunov function for F_u, then x_0 is an asymptoticly stable equilibrium of the system F_u. We will say that there is a polynomial stabilizing control $u(x)$, if there exists $v(x)$ such that $v(x)$ is a strict polynomial Lyapunov function.

Without loss of generality, we assume throughout the paper that $x_0 = e_0$.

3.1 Certificate of Stability

In this section, we will subdivide Δ around the equilibrium and compute a pair (u, v) over a union of subsimplices such that u and v are control and Lyapunov functions for (7). Subdivision of a simplex leads to a local certificate of positivity (for the negative Lie derivative). We suppose that all vector fields are polynomials semidefinite on a union of simplices, $\Delta = W^{[1]} \cup ... \cup W^{[s]}$, as in Figure 2. The graph of a polynomial f over Δ is contained in the *convex hull* of its associated control points. This implies the *range enclosing property* Cargo and Shisha (1966)

$$\min_{|\alpha|=k} b_\alpha(f, k, \Delta) \leq f(x) \leq \max_{|\alpha|=k} b_\alpha(f, k, \Delta),\ x \in \Delta. \tag{8}$$

It follows that the interval (enclosure bound)

$$B(f, k, \Delta) := [\min b_\alpha(f, k, \Delta), \max b_\alpha(f, k, \Delta)]$$

encloses the range of f of degree $l \leq k$ over Δ. Leroy in Leroy (2009) gave convergence properties of the discrete polynomial $f(\sigma_\alpha(k, W^{[i]}))$ to its Bernstein control points (formulas (5) and (6)). We will exploit this convergence to test if the discrete $\mathcal{L}_{F_u}(v)(\sigma_\alpha(k, W^{[i]}))$ is positive semidefinite over $W^{[1]} \cup ... \cup W^{[s]}$.

The proof of the following lemma follows by using the linear convex combinations of the Bernstein coefficients in de Casteljau algorithm Peters (1994).

Lemma 2. Let W be a subsimplex of Δ, which is extracted from Δ by $n+1$ subdivision steps, see 1. Then, by de Casteljau algorithm Peters (1994), the enclosure bound $B_\alpha(f, k, W)$ of f over W is bounded as

$$B(f, k, W) \subseteq B(f, k, \Delta).$$

Lemma 2 illustrates the existence of local certificates of positivity of polynomials in the Bernstein basis under subdivision of a simplex.

Remark 3. Denote the union of the enclosure bounds over $W^{[i]}$, $i = 0, ..., n$, by $B(f, k, W^{[\Delta]})$. For $W^{[0]} \cup ... \cup W^{[n]} \subseteq \Delta$, it holds from Lemma 2 that

$$B(f, k, W^{[\Delta]}) \subseteq B(f, k, \Delta).$$

Example 4. The polynomial $f = 5x^2 - 2x + 1$ is positive on the simplex $\Delta = [-1, 1]$ but the list of Bernstein coefficients $b(f, 2, \Delta) = (8, -4, 4)$. The representation of f in the Bernstein form over $[-1, 1]$ is not sufficient to determine the certificate of positivity. However, by the first binary splitting of Δ, the certificate of positivity of f follows since $b(f, 2, [-1, 0]) = (8, 2, 1)$ and $b(f, 2, [0, 1]) = (1, 0, 4)$.

The following definition from Leroy (2009) will be used for the quadratic convergence (convergence of rate 2) in Theorem 6 and Algorithm 1.

Definition 5. Let $\Delta = [e_0, ..., e_n]$ be a non-degenerate simplex of $\mathbb{R}^n$. For $\gamma = k-2$ and $0 \leq i < j \leq n$, define the second differences of f of degree k with respect to Δ as

$$\begin{aligned}\bigtriangledown^2 b_{\gamma,i,j}(f, k, \Delta) := b_{\gamma+e_i+e_{j-1}} + b_{\gamma+e_{i-1}+e_j} \\ -b_{\gamma+e_{i-1}+e_{j-1}} - b_{\gamma+e_i+e_j},\end{aligned}$$

with the convention $e_{-1} := e_n$. The second differences constitute the collection

$$\bigtriangledown^2 b_{\gamma,i,j}(f, k, \Delta) := (\bigtriangledown^2 b_{\gamma,i,j}(f, k, \Delta))_{|\gamma|=k-2, 0\leq i<j\leq n}.$$

The maximum of the second differences is defined as

$$|| \bigtriangledown^2 b(f, d, \Delta)||_\infty := \max_{|\gamma|=k-2, 0\leq i<j\leq n} | \bigtriangledown^2 b_{\gamma,i,j}(f, k, \Delta)|.$$

Theorem 6. (Leroy, 2009, Theorem 4.9) Let $\Delta = W^{[1]} \cup ... \cup W^{[s]}$ be a subdivision of the standard simplex Δ, and h be an upper bound of the diameters of the $W^{[i]}$'s. Let f be a polynomial function given in the Bernstein form. Then, for $i = 1, ..., s$ and $|\alpha| = k$, we have

$$\max |f(\sigma_\alpha(k, W^{[i]})) - b_\alpha(f, k, W^{[i]})| \leq h^2 S(f),$$

$$S(f) = k \frac{n^2(n+1)(n+2)^2(n+3)}{576} || \bigtriangledown^2 b(f, k, \Delta)||_\infty. \quad (9)$$

The following Remark is from Leroy (2009).

Remark 1. Binary splitting is usually the most efficient subdivision scheme, see Figure 2, regarding the running time as well as the size of the certificates. After at most $n(n+1)/2$ steps of binary splitting of a simplex of diameter h, the diameter of the subsimplices is less than $h/2$.

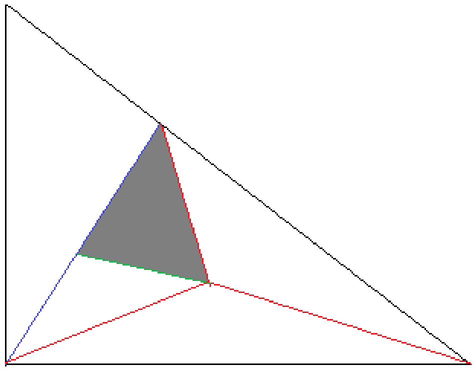

Fig. 1. An extracted subsimplex (shaded) by $n+1$ barycentric subdivision steps.

We will suppose, $\forall |\alpha| = k$, that $\bar{b}^j_\alpha, \bar{b}_\alpha \in \mathbb{R}_{>0}$ and $\underline{b}^j_\alpha \in \mathbb{R}_{<0}$ are picked Bernstein coefficient bounds from the state space. Here, we will not shrink the bounds $[\underline{b}^j_\alpha, \bar{b}^j_\alpha]$ and $[0, \bar{b}_\alpha]$. We subdivide the simplex Δ and compute (u, v) over a union of simplices such that $b_\alpha(\mathcal{L}_{F_u}(v), k, W^{[i]})$ are positive semidefinite

Note that the discrete polynomial $\mathcal{L}_{F_u}(v)(\sigma_\alpha(k, W^{[i]}))$ is unknown. Hence, de Casteljau algorithm is a recursive method for computing the Bernstein coefficients of $\mathcal{L}_{F_u}(v)(\sigma_\alpha(k, W^{[i]}))$ over subsimplices as linear convex combinations of the coefficients over Δ.

Definition 7. (Leroy, 2009, Defenition 5.5) The mesh of $\Delta = W^{[1]} \cup ... \cup W^{[s]}$, denoted by $\hat{m}$, is the largest diameter of $W^{[i]}$, $i = 1, ..., s$. If J is a subdivision scheme, we write $J^N(\Delta)$ the subdivision of Δ obtained after N successive subdivision steps. J is said to have a shrinking factor $0 < C < 1$ if for every simplex Δ, $\hat{m}(J(\Delta)) \leq C\hat{m}(\Delta)$, where $\hat{m}(J(\Delta))$ is the largest mesh among the subsimplices in $J(\Delta)$.

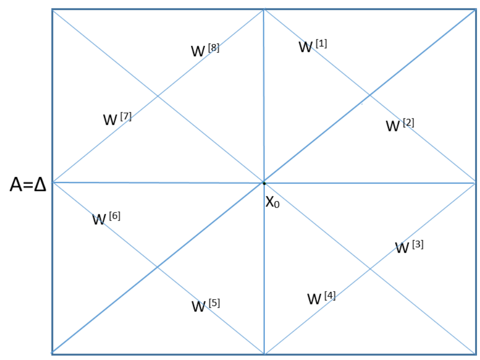

Fig. 2. Binary splitting of a simplex around the equilibrium point.

We let $b_\alpha(v, k, W)$, $b_\alpha(u_j, k, W)$ belong to coefficient bounds $[0, \bar{b}_\alpha]$, $[\underline{b}^j_\alpha, \bar{b}^j_\alpha]$. Consequently, we are able to compute $b_\alpha(\mathcal{L}_{F_u}(v), k, \Delta)$. We will use the assumption in the following corollary to compute a stabilizing control and a Lyapunov function over $W^{[1]} \cup ... \cup W^{[s]}$ so that $b_\alpha(\mathcal{L}_{F_u}(v), k, W^{[i]})$ are positive semidefinite. In the following corollary, we let a small positive δ within the interval spanned between the computed $\min b_\alpha(\mathcal{L}_{F_u}(v), k, \Delta)$ and the minimum of an unknown discrete positive semidefinite

$\mathcal{L}_{F_u}(v)(\sigma_\alpha(k, W^{[i]}))$. If $\mathcal{L}_{F_u}(v)(x)$ satisfies the certificates of positivity over $W^{[i]}$, $i = 1, ..., s$, we say that $\mathcal{L}_{F_u}(v)(x)$ satisfies the local certificate of positivity associated to the subdivision $J(\Delta)$.

Corollary 8. suppose (u, v) are within coefficient bounds $[\underline{b}^j_\alpha, \overline{b}^j_\alpha]$ and $[0, \overline{b}_\alpha]$ such that $\min b_\alpha(\mathcal{L}_{F_u}(v), k, \Delta)$ is non-positive. Let $J(\Delta) = (W^{[1]}, ..., W^{[s]})$ be a subdivision of the simplex Δ, and the interiors of the simplices $W^{[i]}$ are disjoint. Let δ be a small positive real number with $\delta \leq \min \mathcal{L}_{F_u}(v)(\sigma_\alpha(k, W^{[i]}))$. Let N be the number of subdivision steps of Δ, and $C < 1$ is a shrinking factor. Assume that

$$\frac{1}{C^N} > \sqrt{2S/\delta}, \tag{10}$$

where $S = S(\mathcal{L}_{F_u}(v))$ is given by (9). Then, $\mathcal{L}_{F_u}(v)(x)$ satisfies the local certificate of positivity associated to $J^N(\Delta)$.

Proof. Let $\Delta = W^{[1]} \cup ... \cup W^{[s]}$ be a subdivision of Δ so that

$$\begin{aligned} 0 &\leq \delta - \min b_\alpha(\mathcal{L}_{F_u}(v), k, W^{[i]}) \\ &\leq \min \mathcal{L}_{F_u}(v)(\sigma_\alpha(k, W^{[i]})) - \min b_\alpha(\mathcal{L}_{F_u}(v), k, W^{[i]}) \\ &\leq \delta. \end{aligned}$$

Then δ is nonnegative. Theorem 6 implies that

$$\begin{aligned} \min \mathcal{L}_{F_u}(v)(\sigma_\alpha(k, W^{[i]})) - \min b_\alpha(\mathcal{L}_{F_u}(v), k, W^{[i]}) \\ \leq S(\sqrt{2}C^N)^2, \end{aligned}$$

where $\hat{m}(J(\Delta)) = \sqrt{2}$. By rearranging $S(\sqrt{2}C^N)^2 < \delta$ the statement follows. □

In Algorithm 1, we derive from Corollary 8 a condition (10) to test if there is a stabilizing control and a Lyapunov function over a union of simplices. If we satisfy this condition, then we will not need to shrink the coefficient bounds in Algorithm 2.

Algorithm 1. (Stabilizing polynomial control over sub-simplices)

Input. bounds $\overline{b}_\alpha, \overline{b}^j_\alpha \in \mathbb{R}_{>0}$, $\underline{b}^j_\alpha \in \mathbb{R}_{<0}$, for the coefficients of v and u_j, $j = 1, ..., m$, $\delta > 0$, $\Delta = W^{[1]}, ..., W^{[s]}$ with shrinking factor $0 < C < 1$ ($C = 1/2$ say), and N is the number of subdivision steps.

Output. a pair (u, v) with Bernstein coefficients over $W^{[1]} \cup ... \cup W^{[s]}$.

Initialization: $b_\alpha(u_j, k, \Delta) \in [\underline{b}^j_\alpha, \overline{b}^j_\alpha]$, $\forall |\alpha| = k$, $b_\alpha(v, k, \Delta) \in [0, \overline{b}_\alpha]$, with $b_\alpha(v, k, \Delta) = 0$ if $\alpha_0 = k$.

Compute: The constant S, $\min b_\alpha(\mathcal{L}_{F_u}(v), k, W^{[i]})$, $i = 1, ..., s$, with shrinking factor $\frac{1}{C^N} > \sqrt{2S/\delta}$.

1. ***if*** $\min b_\alpha(\mathcal{L}_{F_u}(v), k, W^{[i]}) \geq \delta$

2. ***then*** (u, v) are given within the input bounds over $W^{[1]} \cup ... \cup W^{[s]}$

3. ***else*** proceed to algorithm 2 for shrinking the input bounds

4. ***end if***

Finally, from Algorithm 1, we can compute (u, v) within coefficient bounds over a union of simplices. If there is no stabilizing control within the input coefficient bounds over this union, then we proceed to Algorithm 2 for shrinking the initial coefficient bounds $[\underline{b}^j_\alpha, \overline{b}^j_\alpha]$ and $[0, \overline{b}_\alpha]$.

3.2 Controller Synthesis

We compute a pair (u, v) of polynomial functions within coefficient bounds such that v is a Lyapunov function for F_u. We also estimate Bernstein coefficient bounds for (u, v) over the whole Δ. Specifically, the coefficient bounds of an initial control and a candidate Lyapunov function are recursively shrinked until the Bernstein coefficient bounds of u and v are sufficiently estimated, i.e., Bernstein coefficient bounds for (u, v) such that $\mathcal{L}_{F_u}(v)$ is positive semidefinite.

The Bernstein form of v of degree $l \leq k$ over (without loss of generality) $W \in \{W^{[1]}, ..., W^{[s]}\}$ is given by

$$v = \sum_{|\alpha|=k} b_\alpha(v, k, W) B^{(k)}_\alpha,$$

where $b_\alpha(v, k, W) > 0$ for all $|\alpha| = k$, $\alpha_0 \neq k$. Therefore, as given by Farouki and Rajan Farouki and Rajan (1988)

$$\begin{aligned} v'_i := \frac{\partial v}{\partial x_i}(x) &= \sum_{|\alpha|=k} b_\alpha(v, k, W) \frac{\partial B^{(k)}_\alpha}{\partial x_i}(x) \\ &= \sum_{|\alpha|=k} b_\alpha(v, k, W) k (B^{(k-1)}_{\alpha - e_i} - B^{(k-1)}_\alpha) \\ &\qquad (B^{k-1}_{-1} = B^{k-1}_k = 0) \\ &= \sum_{|\alpha|=k-1} k(b_{\alpha+e_i} - b_\alpha) B^{(k-1)}_\alpha(x), \end{aligned} \tag{11}$$

from which the Bernstein coefficients of v'_i are linear combinations of the coefficients of v.

Remark 9. The number of simplicial Bernstein coefficients of any $n-$dimensional polynomial is $M := \binom{k+n}{k}$.

Define a sub-bound of any $[\underline{b}, \overline{b}]$ by $[\underline{b}_\epsilon, \overline{b}_\epsilon]$, where

$$\underline{b}_\epsilon = \underline{b} + (\overline{b} - \underline{b})\epsilon, \; 0 \leq \epsilon \leq 1, \tag{12}$$

$$\text{and } \overline{b}_\epsilon = \overline{b} - (\overline{b} - \underline{b})\epsilon, \; 0 \leq \epsilon \leq 1. \tag{13}$$

For $t \geq 1$ and $1 > t\epsilon > 0$, we have

$$[\underline{b}, \overline{b}] \supset [\underline{b}_{t\epsilon}, \overline{b}] \supset [\underline{b}_{(t-1)\epsilon}, \overline{b}] \supset .... \supset [\underline{b}_\epsilon, \overline{b}], \; \underline{b}_\epsilon < \overline{b}. \tag{14}$$

Define $L(\epsilon)[\underline{b}, \overline{b}] = [\underline{b}_\epsilon, \overline{b}]$ and $R(\epsilon)[\underline{b}, \overline{b}] = [\underline{b}, \overline{b}_\epsilon]$.

We can also increase the width of any interval bound $[\underline{b}, \overline{b}]$ as

$$\underline{b}^-_\epsilon = \underline{b} - (\overline{b} - \underline{b})\epsilon, \tag{15}$$

and

$$\overline{b}^+_\epsilon = \overline{b} + (\overline{b} - \underline{b})\epsilon, \quad 0 \leq \epsilon \leq 1. \tag{16}$$

Define $L^-(\epsilon)[\underline{b}, \overline{b}] = [\underline{b}^-_\epsilon, \overline{b}]$ and $R^+(\epsilon)[\underline{b}, \overline{b}] = [\underline{b}, \overline{b}^+_\epsilon]$. For the general case, we define

$$Q(\epsilon) = \{L(\epsilon), R(\epsilon), L^-(\epsilon), R^+(\epsilon)\}.$$

Theorem 10 estimates bounds that enclose a Lyapunov polynomial and a polynomial control for (7). The proof provides the choice of subdivision of coefficient bounds used in an algorithm in Section 3.3.

Theorem 10. Let $\overline{b}_\alpha, \overline{b}_\alpha^j \in \mathbb{R}_{>0}$ and $\underline{b}_\alpha^j \in \mathbb{R}_{<0}$, $j = 1, ..., m$, be real numbers. Suppose there exists a stabilizing pair (u, v) for F_u. Then, for $1 \geq \epsilon \geq 0$, there exists $t, s \in \mathbb{N}^+$ and $d_1^j, ..., d_t^j \in Q(\epsilon)$ with $q_1, ..., q_s \in Q(\epsilon)$ such that

$$b_\alpha(u_j, k, W) \in d_t^j \circ d_{t-1}^j \circ ... \circ d_1^j[\underline{b}_\alpha^j, \overline{b}_\alpha^j] \text{ and}$$
$$b_\alpha(v, k, W) \in q_s \circ q_{s-1} \circ ... \circ q_1[0, \overline{b}_\alpha].$$

Proof. Let $b_\alpha(v, k, W) \in [0, \overline{b}_\alpha]^M$ for all $|\alpha| = k$, where $b_\alpha(v, k, W) = 0$ for $\alpha_0 = k$, and $b_\alpha(u_j, k, W) \in [\underline{b}_\alpha^j, \overline{b}_\alpha^j]^M$ for all $|\alpha| = k$ and denote this control by $u(x)$. It is sufficient to let the coefficients are belong to the vertices of the boxes $[\underline{b}_\alpha^j, \overline{b}_\alpha^j]^M$, $[0, \overline{b}_\alpha]^M$.

The Bernstein coefficients of $\mathcal{L}_{F_u}(v)(x)$ can be rearranged as (cf. Definition 1, where $b_\alpha^j(u)$ stands for $b_\alpha(u_j, k, W)$ and $b_\alpha(v)$ for $b_\alpha(v, k, W)$)

$$\begin{aligned} b_\alpha(\mathcal{L}_{F_u}(v), k, W) &= \sum_{i=1}^n p_i(b_\alpha(v))_i + b_\alpha^1(u_1) \sum_{i=1}^n g_{i1}(b_\alpha(v))_i \\ &+ ... + b_\alpha^m(u_m) \sum_{i=1}^n g_{im}(b_\alpha(v))_i, \ |\alpha| = k. \quad (17) \\ &=: H_\alpha(v) + H_\alpha(u_1, v) + ... + H_\alpha(u_m, v). \end{aligned}$$

Hence,

$$\begin{aligned} \mathcal{L}_{F_u}(v)(x) &= \sum_{|\alpha|=k} H_\alpha(v) B_\alpha(x) + \sum_{|\alpha|=k} H_\alpha(v, u_1) B_\alpha(x) \\ &+ ... + \sum_{|\alpha|=k} H_\alpha(v, u_m) B_\alpha(x). \end{aligned}$$

Let $\min b_\alpha(\mathcal{L}_{F_u}(v), k, W) =: H_{\alpha^*}(v) + H_{\alpha^*}(u_1, v) + ... + H_{\alpha^*}(u_m, v)$, for some $|\alpha^*| = k$ is positive. Then

$$\mathcal{L}_{F_u}(v)(x) > 0, \ \forall x \in W \setminus \{e_0\}, \quad (18)$$

and (u, v) have Bernstein coefficients within $[0, \overline{b}_\alpha]$ and $[\underline{b}_\alpha^j, \overline{b}_\alpha^j]$, $j = 1, ..., m$, $\epsilon = 0$. Otherwise, suppose $\underline{\mathcal{L}}_{F_u}(v)$ is non-positive. Then for $\epsilon > 0$, we have the following cases.

Case 1: $H_{\alpha^*}(u_{j_0}, v) \leq 0$ ($H_{\alpha^*}(v) > 0$), for some $j_0 \in \{1, ..., m\}$. In this case, we apply $R(\epsilon)[\underline{b}_{\alpha^*}^{j_0}, \overline{b}_{\alpha^*}^{j_0}]$ if the coefficients of u_{j_0} are positive (we apply $L(\epsilon)[\underline{b}_{\alpha^*}^{j_0}, \overline{b}_{\alpha^*}^{j_0}]$ if the coefficients of u_{j_0} are non-positive). This shrinking of bounds of some u_{j_0} does not affect the other positive terms of (17). For $\epsilon > 0$, we deduce

$$H_{\alpha^*}(u_{j_0}^*, v) > H_{\alpha^*}(u_{j_0}, v),$$

where $u_{j_0}^*$ denotes the control with coefficients belong to $R(\epsilon)[\underline{b}_{\alpha^*}^{j_0}, \overline{b}_{\alpha^*}^{j_0}]$.

Case 2: $H_{\alpha^*}(v) \leq 0$ ($H_{\alpha^*}(u_j, v) > 0$, $\forall j$). We apply $R(\epsilon)[0, \overline{b}_{\alpha^*}]$. Here, shrinking the bound of v decreases the other positive terms of (17). Hence, we apply $R^+(\epsilon)[\underline{b}_{\alpha^*}^j, \overline{b}_{\alpha^*}^j]$, $\forall j = 1, ..., m$, if the coefficients of u_j in (17) are positive (If the coefficients of some u_j are non-positive we apply $L^-(\epsilon)[\underline{b}_{\alpha^*}^j, \overline{b}_{\alpha^*}^j]$). This means, we shrink the bound of v and increase the corresponding independent bounds of u_j by the value ϵ.

Case 3: $H_{\alpha^*}(v) \leq 0$ and $H_{\alpha^*}(u_{j_0}, v) \leq 0$, for some $j_0 \in \{1, ..., m\}$. In this case, we apply $R(\epsilon)[0, \overline{b}_{\alpha^*}]$ and $R(\epsilon)[\underline{b}_{\alpha^*}^{j_0}, \overline{b}_{\alpha^*}^{j_0}]$, if the corresponding coefficients of u_{j_0} are positive (for the non-positive coefficients of u_{j_0} we apply $L^-(\epsilon)[\underline{b}_{\alpha^*}^{j_0}, \overline{b}_{\alpha^*}^{j_0}]$). Furthermore, we apply $R^+(\epsilon)[\underline{b}_{\alpha^*}^j, \overline{b}_{\alpha^*}^j]$, $\forall j = 1, ..., m$, $j \neq j_0$. This holds

$$H_{\alpha^*}(v^*) + H_{\alpha^*}(u_{j_0}^*, v^*) > H_{\alpha^*}(v) + H_{\alpha^*}(u_{j_0}, v),$$

where v^* denotes a Lyapunov polynomial of coefficients belong to $R(\epsilon)[0, \overline{b}_{\alpha^*}]$. The restriction of all cases above, for all $i = 1, .., \max\{s, t\}$, is that $\overline{b}_{\alpha^*}(v) - i\epsilon \geq 0$. Hence, we may compute some $d_t^{j_0} \circ d_{t-1}^{j_0}[\underline{b}_{\alpha^*}^{j_0}, \overline{b}_{\alpha^*}^{j_0}] = 0$. This eliminate the corresponding j_0-th term of $\underline{\mathcal{L}}_{F_u}(v)$. It follows that $\underline{\mathcal{L}}_{F_u^*}(v)$ is positive with coefficients for (u, v) belong to $q_s \circ q_{s-1}[0, \overline{b}_\alpha]$ and $d_t^j \circ d_{t-1}^j[\underline{b}_\alpha^j, \overline{b}_\alpha^j]$. Repeating the arguments for all $|\alpha| = k$ that refer to all non-positive coefficients of $b_\alpha(\mathcal{L}_{F_u}(v), k, W)$, the proof follows. □

3.3 Algorithm for Controller Synthesis

In this section, we suppose that there exist a stabilizing control and a Lyapunov function for the system (7). Subsequently, we derive from Theorem 10 an algorithm that computes (u, v) within coefficient bounds so that $\underline{\mathcal{L}}_{F_u}(v)$ (Remark **??**) is positive semidefinite. Specifically, we will approximate bounds by shrinking or increasing the width of Bernstein bounds of u and v.

Let

$$\mathcal{L}_{F_u}(v)(x) = (\frac{\partial v}{\partial x}(x))^T (p(x) + g(x) u(x)),$$

where $u : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is the control polynomial function. For $k \leq l$ and $W \in \{W^{[0]}, ..., W^{[n]}\}$, the Bernstein coefficients $b_\alpha(\mathcal{L}_{F_u}(v), k, W)$ are given by (17). It follows that, $\forall x \in W$,

$$\min b_\alpha(\mathcal{L}_{F_u}(v), k, W) \leq \mathcal{L}_{F_u}(v)(x) \leq \max b_\alpha(\mathcal{L}_{F_u}(v), k, W).$$

Rearrange the Bernstein coefficients of $\mathcal{L}_{F_u}(v)(x)$

$$\begin{aligned} b_\alpha(\mathcal{L}_{F_u}(v), k, W) &= \sum_{i=1}^n p_i(b_\alpha(v))_i + b_\alpha^1(u_1) \sum_{i=1}^n g_{i1}(b_\alpha(v))_i \\ &+ ... + b_\alpha^m(u_m) \sum_{i=1}^n g_{im}(b_\alpha(v))_i \\ &= H_\alpha(v) + H_\alpha(u_1, v) + ... + H_\alpha(u_m, v), \end{aligned}$$

and suppose

$$\begin{aligned} &\underline{\mathcal{L}}_{F_u}(v) = b_{\alpha^*}(\mathcal{L}_{F_u}(v), k, W) \\ &= H_{\alpha^*}(v) + H_{\alpha^*}(u_1, v) + ... + H_{\alpha^*}(u_m, v), \text{ for some } |\alpha^*| = k. \quad (19) \end{aligned}$$

Algorithm 2. (Computing controller within coefficient bounds)

Input. $\underline{b}_\alpha^j \in \mathbb{R}_{<0}$, $\overline{b}_\alpha^j, \overline{b}_\alpha \in \mathbb{R}_{>0}$, $j = 1, ..., m$, and $0 < \epsilon$, with $t, s \in \mathbb{N}_{>0}$ and $Q^j(\epsilon)[\underline{b}_\alpha^j, \overline{b}_\alpha^j]$, $Q(\epsilon)[0, \overline{b}_\alpha]$.

Output $d_t^j \circ d_{t-1}^j \circ ... \circ d_1^j[\underline{b}_\alpha^j, \overline{b}_\alpha^j]$, $j = 1, ..., m$, and $R(s\epsilon)[0, \overline{b}_\alpha]$.

Initialization: $b_\alpha(u_j,k,W) \in [\underline{b}^j_\alpha, \overline{b}^j_\alpha]^M =: d^j_0[\underline{b}^j_\alpha, \overline{b}^j_\alpha]^M$, $b_\alpha(v,k,W) \in [0, \overline{b}_\alpha]^M =: R(0)[0,\overline{b}_\alpha]^M$, with $b_\alpha(v,k,W) = 0$ if $\alpha_0 = k$.

Compute: (the minimum of (17)) $\underline{\mathcal{L}}_{F_u}(v) = H_{\alpha^*}(v) + H_{\alpha^*}(u_1,v) + ... + H_{\alpha^*}(u_m,v)$, for some $|\alpha^*| = k$.

1. ***if*** $\underline{\mathcal{L}}_{F_u}(v) > 0$
2. ***then*** $\mathcal{L}_{F_u}(v)(x) > 0$
3. ***else if*** $\underline{\mathcal{L}}_{F_u}(v) \le 0$
4. ***then*** for $i = 1, ..., \max\{t,s\}$ ***do*** compute
$$d^j_i \circ d^j_{i-1}[\underline{b}^j_\alpha, \overline{b}^j_\alpha], R(i\epsilon)[0, \overline{b}_\alpha]$$
5. **end** ***if***
6. **end** ***if***
7. **return** $\underline{\mathcal{L}}_{F^*_u}(v)$ with d^j_i, $R(i\epsilon)$.

Algorithm 2 tests if there is a stabilizing control and a Lyapunov function within bounds $[\underline{b}^j_\alpha, \overline{b}^j_\alpha]$, $[0, \overline{b}_\alpha]$, and computes $d[\underline{b}^j_{\alpha^*}, \overline{b}^j_{\alpha^*}]$ and $q[0, \overline{b}_{\alpha^*}]$ such that $\mathcal{L}_{F_u}(v)(x)$ is positive semidefinite.

Lemma 11. Under the assumptions of Theorem 10, the number of iterations (t,s) needed to compute $d^j_t \circ d^j_{t-1} \circ ... \circ d^j_1[\underline{b}^j_{\alpha^*}, \overline{b}^j_{\alpha^*}]$, $j = 1, ..., m$, and $q_s \circ q_{s-1} \circ ... \circ q_1[0, \overline{b}_\alpha]$, for some $|\alpha^*| = k$, that yield a negative $\underline{\mathcal{L}}_{F_u}(v)$ is $\max\{t,s\} \le \overline{b}_{\alpha^*}$, where $\overline{b}_{\alpha^*}$ denotes the upper coefficient bound of v.

Proof. Suppose $b_{\alpha^*}(v,k,W) \in [0, \overline{b}_{\alpha^*}(v)]$ for some $|\alpha^*| = k$, and $b_{\alpha^*}(u_j,k,W) \in [\underline{b}^j_{\alpha^*}(u), \overline{b}^j_{\alpha^*}(u)]$, $j = 1, ..., m$. From (17), let the cases 1-3 above are hold. Then, we deduce from $R(i\epsilon)[0, \overline{b}_{\alpha^*}]$ of v that $\overline{b}_{\alpha^*}(v) - i\epsilon \ge 0$, where i denotes the loop indicator of $i\epsilon \ge 0$. It follows for $i = 1, ..., \max\{s,t\} =: z$,
$$0 \le \overline{b}_{\alpha^*}(v) - z\epsilon \le \overline{b}_{\alpha^*}(v) - (z-1)\epsilon \le ... \le \overline{b}_{\alpha^*}(v) - \epsilon,$$
from which $z\epsilon \le \overline{b}_{\alpha^*}(v)$. Since increasing the width of the bounds of the corresponding u_j in (17) depends on the same ϵ, the statement follows. □

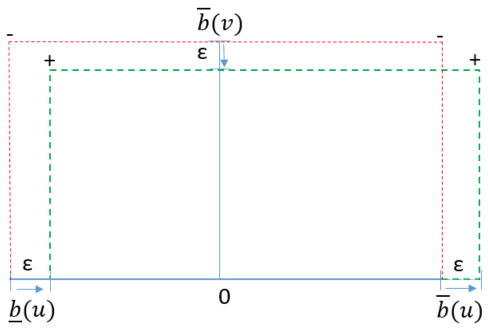

Fig. 3. Shrinking the upper bound of v, shrinking the lower bound of u and raising its upper bound.

Finally, we guarantee that $\mathcal{L}_{F_u}(v)(x) > 0$ for all $x \in W \setminus \{e_0\}$ by degree elevation or subdivision of W. Degree elevation satisfies the global certificates of positivity. Repeatedly subdivision of W satisfies local certificates of positivity, whereas, Algorithm 2 computes the best coefficient bounds for u and v with few zero bounds.

To illustrate Algorithm 2, we compute on coefficient in the following example for (u,v) such that the negative Lie derivative is positive semidefinite.

Example 12. Let
$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{pmatrix} = \begin{pmatrix} x_1^3 \\ x_1 - 8x_2^2 \\ -x_1x_2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$
be of degree $l = 3$ over Δ. In order to compute $v(x)$ and $u(x)$, we may suppose $[0, \overline{b}^v_\alpha]^M = [0,3] \times [0,10]^8 \times [0,5]$, and $[\underline{b}^u_\alpha, \overline{b}^u_\alpha]^M = [-5,5] \times [-4,2]^8 \times [-6,1]$. We give the method, for simplicity, for some $|\alpha^*| = l$, i.e., $b_{\alpha^*}(v,l,V) = \overline{b}^v_{\alpha^*}$ and $b_{\alpha^*}(u,l,V) = \overline{b}^u_{\alpha^*}$. Let $b_{\alpha^*}(v,l,V) = 10$ and $b_{\alpha^*}(u,l,V) = 2$, and compute $(\epsilon = 0)$
$$b_{\alpha^*}(\mathcal{L}_{F_u}(v)) = H_{\alpha^*}(v) + H_{\alpha^*}(u_1,v) + H_{\alpha^*}(u_2,v)$$
$$= -70 + 2(10) + 2(10) = -30. \text{ (negative)}$$
Note that $H_{\alpha^*}(v) < 0$ and $H_{\alpha^*}(u,v) > 0$. Hence, we let $\epsilon = 1/4$ and apply $R(1/4)[0,10] = [0,8]$ and increase the corresponding upper bound of u, case 2 above, $R^+(1/4)[-4,2] = [-4,3.5]$. Computing $b_{\alpha^*}(\mathcal{L}_{F^*_u}(v))$ with the new bounds of (u,v) result
$$b_{\alpha^*}(\mathcal{L}_{F^*_u}(v)) = -56 + 3.5(8) + 3.5(8) = 0.$$
Eventually, at the second iteration $(2\epsilon = 1/2)$ of subdivision bounds, $b_{\alpha^*}(\mathcal{L}_{F^*_u}(v))$ is negative with $R(1/2)[0,10] = [0,4]$ and $R^+(1/2)[-4,2] = [-4,5]$. It follows that the number of iterations $(t = s)$ needed to (implement Algorithm 2) increase and shrink the bounds is equal 2, where $b_{\alpha^*}(v) = 4$ and $b_{\alpha^*}(u) = 5$. Repeatedly applying the algorithm for all $|\alpha| = l$, computes all coefficients of u and v. Subsequently, we have $-\frac{\partial v}{\partial x}(x)F_u(v)(x) \ge \underline{\mathcal{L}}_{F_u}(v), \forall x \in V \setminus \{e_0\}$, from which the Bernstein coefficients of (u,v) are belong to the estimated bounds.

4. CONCLUSIONS

In this paper, we exploited certificates of positivity in the Bernstein basis for polynomial control systems. We investigated certificates of positivity in the simplicial Bernstein basis by degree elevation and subdivision. This satisfied the stability of the designed feedback system in the Bernstein form over a union of simplices. Subsequently, we developed an algorithm for computing a polynomial Lyapunov function and a control polynomial function within Bernstein coefficient bounds. Finally, we provided a new strategy for estimating the coefficient bounds of u and v.

ACKNOWLEDGEMENTS

This work is supported by the Danish Council for Independent Research under grant number DFF- 4005-00452 in the project CodeMe.

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